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Calc 3 18.10.21

Ex Find the points on the sphere $x^2+y^2+z^2=4$ closest to and furthest from $(3,-1,1)$
↳ Trying to optimize distance subject to sphere

optimize $f(x,y,z)=(x-3)^2+(y+1)^2+(z-1)^2$
subject to $x^2+y^2+z^2=4$

$$f(x,y,z) = x^2-6x+9+y^2+2y+1+z^2-2z+1$$
$$(x^2+y^2+z^2) = (-11)+6x-2y+2z$$

So we will work with
$$F(x,y,z) = f(x,y,z) - \lambda g(x,y,z)$$
$$\nabla F = \langle -6-2\lambda x, 2-2\lambda y, 2-2\lambda z, -(x^2+y^2+z^2-4)\rangle$$

So $\nabla F = \vec{0}$ iff
$$\begin{cases} -6-2\lambda x = 0 \\ 2-2\lambda y = 0 \\ 2-2\lambda z = 0 \\ x^2+y^2+z^2-4 = 0 \end{cases}$$

Observe $\lambda = \pm\frac{\sqrt{11}}{2}$

$\lambda^2(x^2+y^2+z^2) = \lambda^2 4$
$-3^2+1^2-1^2 = 4^2 \rightarrow$ (substituting $(2,2,3)$)

Now, there is two cases (2 pts)

$\lambda = \frac{+\sqrt{11}}{2}$, solve for x,y,z
$A = \left\langle \frac{-6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}} \right\rangle$

$\lambda = -\frac{\sqrt{11}}{2}$
$B = \left\langle \frac{+6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}} \right\rangle$

$f(A) = 15 - 6\frac{-6}{\sqrt{11}} + 2\left(\frac{2}{\sqrt{11}}\right) + 2\left(\frac{2}{\sqrt{11}}\right)$
$= 15+\frac{44}{\sqrt{11}}$

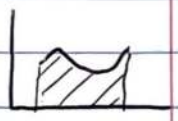
$f(B) = 15 - 6\frac{+6}{\sqrt{11}} - \frac{4}{\sqrt{11}} + \frac{4}{\sqrt{11}}$
$= 15 - \frac{36}{\sqrt{11}}$

$f(A)$ is max distance^2 and $f(B)$ is min distance^2 by Legrange Multipliers. A is furthest, B is closest

Exercise: A box (without lid) is to be built with surface area 12. What's the maximum volume?

Double Integrals (15.1)

Idea ~ We have functions of several variables. What should it mean to integrate them?



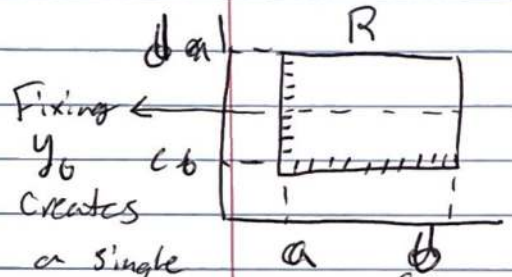
In Calc I: Area under the curve / antiderivitive

In Calc III: The definite integral of f over region R should be the "net volume" of f over R

$$\iint_R f\,dA = \text{volume}$$

double definitive → $\iint_R$ ← region; dA ← respect to Area

Today → R is as simple as possible (R is a rectangle)



$$R = [a,b][c,d] = \{(x,y) \mid x \in [a,b], y \in [c,d]\}$$

Using the same trick from calc I, we can approx. the volume over R by using left lower corner points to determine height of boxes via f(corner point)

Goal: use calculus I to solve $\iint_R f\,dA$

$$\iint_R f\,dA \to \int_x \int_y$$

Theo Prop ~ Fubini's Theorem: If $f(x,y)$ is cont on rectangle $R=[a,b]\times[c,d]$, then

$$\int_{y=c}^{d}\left(\int_{x=a}^{b} f(x,y)\,dx\right)dy = \iint_R f\,dA$$

(also works in other direction)

Ex. Compute $\iint_R x\sec^2(y)\,dA$ for $R=[0,2]\times[0,\frac{\pi}{4}]$

By Fubini's Theorem

$$\int_{y=0}^{\frac{\pi}{4}}\left(\int_{x=0}^{2} x\sec^2(y)\,dx\right)dy$$

$$=\int_{y=0}^{\frac{\pi}{4}} \sec^2(y)\left(\int_{x=0}^{2} x\,dx\right)dy$$

$$=\int_{y=0}^{\frac{\pi}{4}} 2\sec^2 y\,dy$$

$$=2\left[\tan y\right]_0^{\frac{\pi}{4}} = 2$$

ex. Compute $\iint_R \frac{1}{1+x+y}\,dA$ for $R = [1,2] \times [2,3]$

$$\int_{x=1}^{2}\int_{y=2}^{3} \frac{1}{1+x+y}\,dy\,dx \qquad \begin{array}{l} u = 1+x+y \\ du = 1\,dy \end{array}$$

$$\int \frac{1}{u}\,du\,dx$$

$$\int_{x=1}^{2} \ln(|4+x|) - \ln(|3+x|)\,dx$$

$$= (4+x)\ln(4+x) - (3+x)\ln(3+x) \Big|_{y=1}^{2}$$